MODELLING HIERARCHICAL KEY STRUCTURE WITH PITCH SCAPES

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ABSTRACT

Musical form and syntax in Western classical music are hierarchically organised on different timescales. One of the most important features of this structure is the organisation of modulations between different keys throughout a piece. Music theoretical research has established taxonomies of prototypical modulation plans for different modes and musical forms. However, these prototypes still require empirical validation based on quantitative statistical methods and cannot be retrieved automatically so far.

In this paper, we present a novel method to infer prototypical modulation plans from musical corpora. A modulation plan is formalised as a transposition-invariant probabilistic model over the underlying pitch class distributions based on a hierarchical *pitch scape* representation. Prototypical modulation plans can be learned in an unsupervised manner by training a mixture model (similar to a Gaussian mixture model) on the data, so that different prototypes appear as distinct clusters.

We evaluate our approach by performing hierarchical clustering on a corpus of more than 150 Baroque pieces, with the extracted clusters showing excellent agreement with the most common prototypes postulated in music theory. Our method bears a great potential for modelling, analysis and discovery of hierarchical key structure and prototypes in corpora across a broad range of musical styles. An accompanying library is available at: `github.com/robert-lieck/pitchscapes`.

1. INTRODUCTION

The hierarchical structure of a piece in Western classical music is strongly determined by musical form [1] and harmonic syntax [2, 3], based on different aspects, such as repetition and variation of the rhythmic, melodic and harmonic content and hierarchical relations between different harmonies.

A central aspect that links musical form and harmonic syntax is the modulation plan of a piece. Western musicology assumes a number of prototypical modulation plans that describe the overarching tonal structure of a piece, such as I–V–I for pieces in major or i–III–i for pieces in minor [1]. These prototypes have a long-standing history in musicology and have emerged from inspection of numerous individual pieces and agreement among experts. However, a quantitative validation based on statistical methods constitutes an important supplement to confirm and refine the music theoretic findings. Furthermore, they cannot be automatically retrieved from musical data, which impedes large-scale investigations and the application to other styles and genres of music.

In this paper, we present a method to retrieve prototypical modulation plans from large corpora of musical pieces in an unsupervised manner. This is achieved by modelling the overall corpus as a mixture of multiple prototypes, similar to how Gaussian mixture models [4] can be applied to clustering in Euclidean space. A prototype is represented by a transposition-invariant Bayesian model that describes the pitch content of a piece (pitch class distributions) on multiple time scales. Modelling is based on a novel *pitch scape* representation of the musical content, which allows to account for the hierarchical structure inherent to both musical form and harmonic syntax. We evaluate our model on a corpus of more than 150 Baroque pieces, with the extracted clusters showing excellent agreement with the most common prototypes postulated in music theory.

By providing a solid statistical approach to modelling prototypical modulation plans, we make an important contribution to connecting music theory and empirical science. Our approach relies on minimal prior assumptions, works on simple pitch data, and learns prototypes in an unsupervised manner, which bears a great potential for modelling, analysis and discovery of hierarchical key structure and prototypes in corpora across a broad range of musical styles.

In the remainder of the paper, we describe the underlying *pitch scape* representation in Section 2, introduce the probabilistic Bayesian model that is used to learn prototypes and prototype mixtures from musical corpora in Section 3, and present and discuss the results of our evaluation in Section 4.

2. PITCH SCAPES

We model prototypical modulation plans based on a novel *pitch scape* representation of the musical content. Pitch scapes (see Figure 1 for an illustration) represent the pitch content of a piece on multiple time scales and can be for-

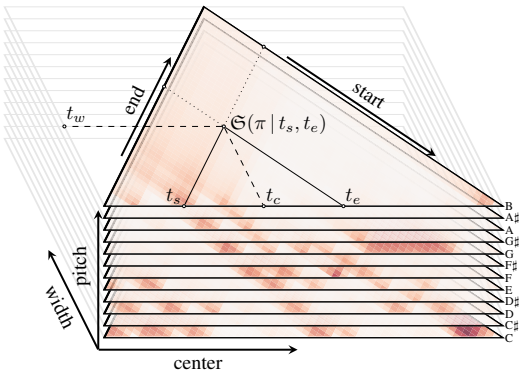

Figure 1. Pitch scape (Prelude in C major, BWV 846, Johann Sebastian Bach). The two time values can be specified in start-end-coordinates (t_s and t_e) or in center-width-coordinates (t_c and t_w).

mally defined as the conditional probability distribution of the pitch classes for a given section of the piece:

Definition 1 (Pitch Scape)**.** *A pitch scape $\mathfrak{S}$ is a function that maps each proper time interval $[t_s, t_e]$ ($t_s < t_e$) to a pitch class distribution*

$$\mathfrak{S} : \mathbb{R} \times \mathbb{R} \to [0,1]^{12} , \quad \sum_{\pi=0}^{11} \mathfrak{S}(\pi \mid t_s, t_e) = 1 . \qquad (1)$$

A pitch scape can equivalently be conceived as a conditional probability distribution $\mathfrak{S}(\pi \mid t_s, t_e)$ with three variables or a vector-valued function $\mathfrak{S}(t_s, t_e)$ in two variables.

Pitch scapes are inspired by scape plot visualisations, to which we draw the connection in Section 2.1, while Section 2.2 describes how to compute pitch scape estimates for a given piece.

2.1 Pitch Scape Visualisation

Scape plot visualisations were introduced in [5,6] to depict key estimates for different sections of a piece in a hierarchical triangular plot and have since been used for a variety of visualisation tasks [7–11].

Visualising the entire information contained in a three-dimensional pitch scape in a single two-dimensional plot is difficult. However, there are two convenient ways to visualise the relevant information. First, the 12 components can be visualised separately by creating one scape plot per pitch class. This preserves the entire information but does not foster musical intuition because information about simultaneous events is scattered across multiple plots. Alternatively, a key finding algorithm can be employed [12–15] to map the pitch class distribution of each point to a colour value. This corresponds to a key-scape plot of the pitch scape. For illustration, we show in Figure 2 an overlay of the 12 separate pitch-scape plots and the corresponding key-scape plot.

The colour mapping for a key-scape plot can be realised in different ways. We use a template-based key finder that

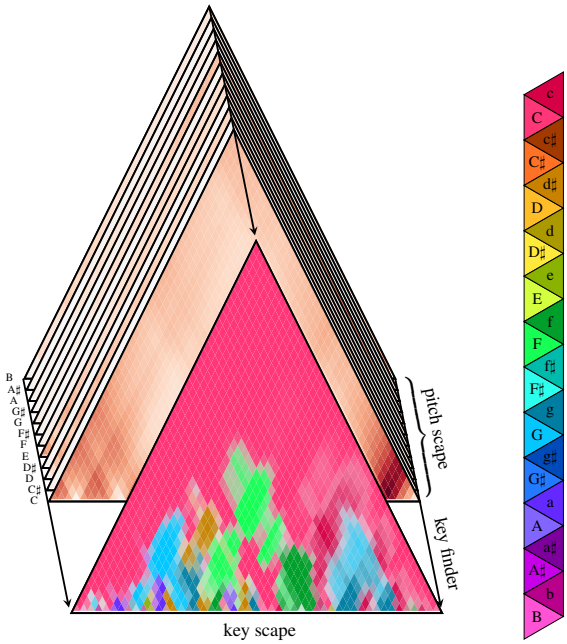

Figure 2. Separate pitch-scape plots and resulting key-scape plot for the prelude in C major, BWV 846, Johann Sebastian Bach (colour legend for keys on the right).

provides a score value for each major and minor key. The scores can be transformed into a probability distribution $p(k)$ using a soft-max function. After choosing a unique colour for each key, $p(k)$ can be used to interpolate between colours by computing their weighted average. To define the colour for each key, we let the hue value vary either along the circle of fifths or chromatically, which has complementary advantages. Fifth-based hue maps related keys to similar colours, while chromatic hue allows to better distinguish them. We add a lightness offset to distinguish major and minor keys and map the entropy of $p(k)$ to saturation. Entropy-based saturation allows to indicate regions with uncertain key classification and avoids uninterpretable colour blends.

2.2 Pitch Scape Estimates

The pitch scape of a piece is computed from its musical content and reflects the probability of a certain pitch class to occur within the specified time interval. As we are working in a Bayesian framework, we model the pitch scape of a piece as a posterior estimate given a prior distribution and the observed notes. To formally define the pitch scape estimate of a piece, we first define its pitch class density:

Definition 2 (Pitch Class Density)**.** *The pitch class density $\delta(\pi \mid t)$ for pitch class π at time t corresponds to the normalised pitch class counts over all tones that sound at time t*

$$\delta(\pi \mid t) := \frac{1}{\max\{1, |T_t|\}} \sum_{\tau \in T_t} [\![\tau \bmod 12 = \pi]\!] , \qquad (2)$$

where T_t is the multiset of all tones (as integers in MIDI pitch representation) sounding at time t; $[\![\cdot]\!]$ is the Iverson

bracket, which equals 1 if its argument is true and 0 otherwise; and the max *avoids division by zero for silent parts where $T_t = \emptyset$ is the empty set.*

Using the pitch class density, we define the pitch scape estimate as follows:

Definition 3 (Pitch Scape Estimate)**.** *The posterior estimate of the pitch scape $\mathfrak{S}(\pi \,|\, t_s, t_e)$ for pitch class π and time interval $[t_s, t_e]$ is*

$$\mathfrak{S}(\pi \,|\, t_s, t_e) := \underbrace{\frac{1}{t_e - t_s + 12c}}_{\text{normalisation}} \Big[\overbrace{c}^{\text{prior counts}} + \underbrace{\int_{t_s}^{t_e} \delta(\pi \,|\, t)\, dt}_{\text{overall pitch class counts}} \Big] \,, \quad (3)$$

where the integral over the pitch class density computes the overall pitch class counts, $c \geq 0$ specifies the prior counts, and the leading term ensures proper normalisation.

Using zero prior counts $c = 0$ thus corresponds to using the average pitch class density as pitch scape estimate (in Bayesian terms this would be a maximum likelihood estimate). In contrast, using a prior count of $c = 1$, which is done throughout the paper, corresponds to a Bayesian maximum posterior estimate with a uniform prior over pitch classes ($c = 1$ corresponds to a uniform Dirichlet distribution, which is the appropriate conjugate prior for the categorical distribution over pitch classes). Note that choosing $c > 0$ also circumvents the zero-count problem for silent parts.

The relative weight of the overall pitch class counts, computed in the integral in (3), depends on the scale on which time is measured. Throughout the paper, we measure time in quarter notes, so that a time interval of one quarter note has the weight of a single observation. That means, for instance, a single pitch sustained for two quarter note adds two to the respective overall pitch class counts; two different pitches sustained for one quarter note add half a count each; and three pitches sustained for an eight note add one sixth count each. Thus, for small time intervals the prior counts c introduce a significant bias towards a uniform pitch class distribution, while for large time intervals they have a vanishingly small weight relative to the overall pitch class counts (e.g. a 32-bar piece in 4/4 yields a total of 128 pitch class counts, so that the prior counts do not cause a major change of the estimated pitch class distribution for the entire piece).

3. MODELLING KEY STRUCTURE

We define our model for mixtures of prototypes in two steps. First, we define a probabilistic pitch scape model of a transposition-invariant modulation plan (Section 3.1). Based on this model for single prototypes, we define a mixture model (Section 3.2) that incorporates explicit transposition and models a musical corpus as a mixture of multiple prototypes.

3.1 Prototypes

The idea of a prototype is to specify an object that represents a subset of the data. In probabilistic modelling this corresponds to defining a probability distribution for which a subset of the data has a high likelihood. Additionally, this distribution should be unimodal so that its mode can be taken as a representative of all data points belonging to that prototype. In n-dimensional Euclidean space, prototypes can for instance be defined using multivariate Gaussian distributions.

When defining prototypes for pitch scapes, we are facing some additional challenges that will be addressed in the following. First, the output of a pitch scape is a categorical distribution (over pitch classes), which has to be normalised. Second, time is continuous so that a pitch scape itself is an inherently continuous object. Third, the prototypes postulated in music theory are formulated in terms of scale degrees, which makes them transposition-invariant (i.e. transposing a piece does not affect its relation to a specific prototype). The first two points will be addressed in the following Section 3.1.1, the third point is resolved in Section 3.1.3 and incorporated in the mixture model in Section 3.2.

3.1.1 Definition

We address the first two points by defining a prototype as a point-wise Dirichlet distribution with a time-dependent parameter vector α. The Dirichlet distribution is the conjugate prior of a categorical distribution and acts as a likelihood function if the observations themselves are categorical distributions, as it is the case for individual points in a pitch scape (first point). Making its parameter vector time-dependent additionally allows it to vary continuously over the pitch scape (second point). Formally, a prototype is defined as follows:

Definition 4 (Prototype)**.** *Given a function*

$$\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+^{12} \quad (4)$$

that maps each proper time interval $[t_s, t_e]$ ($t_s < t_e$) to a vector with positive entries, a prototype is defined as the point-wise Dirichlet distribution with parameter vector α. The likelihood of observing a pitch class distribution Π for the interval $[t_s, t_e]$ given α is

$$p(\Pi \,|\, \alpha, t_s, t_e) = \mathrm{Dir}(\Pi; \alpha(t_s, t_e)) \,. \quad (5)$$

The log-likelihood of observing a full pitch scape $\mathfrak{S}$ given α is

$$\log p(\mathfrak{S} \,|\, \alpha) = \frac{2}{T^2} \iint\limits_{0 \leq t_s < t_e \leq T} \log \mathrm{Dir}\big(\mathfrak{S}(t_s, t_e); \alpha(t_s, t_e)\big) dt_s dt_e \,, \quad (6)$$

where T is the duration of the piece.

The definition of the log-likelihood in (6) is equivalent to the (negative) cross-entropy of an infinite number of uniform samples from the pitch scape. It differs from a simple integration of (5) only by the normalisation $\frac{2}{T^2}$, which rescales it to the magnitude of a single observation and makes it invariant to the duration of the piece. Note that both (5) and (6) are probability *density* functions with the usual implications (i.e. they can be greater than 1; their log can be positive; their cross-entropy can be negative).

3.1.2 Proxy Function

To learn prototypes from data, we define α via a three dimensional real-valued proxy function $\widetilde{\alpha}$ that has a set of adjustable parameters θ and an open parameter τ

$$\widetilde{\alpha}^{(\theta,\tau)} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \,. \tag{7}$$

The domain of interest is $[0,1] \times [0,1] \times \mathbb{Z}_{12}$ with the first two arguments specifying the time interval in normalised center-width-coordinates and the third specifying the pitch class. The π^{th} component of α is then defined to be

$$\alpha_{\pi}^{(\theta,\tau)}(t_s, t_e) := e^{\widetilde{\alpha}^{(\theta,\tau)}(\bar{t}_c, \bar{t}_w, \pi)} \tag{8}$$

with

$$\bar{t}_c = \tfrac{1}{2T}(t_s + t_e) \qquad \bar{t}_w = \tfrac{1}{T}(t_e - t_s) \,, \tag{9}$$

where T is the duration of the piece that is to be modelled.

3.1.3 Fourier Representation

We parameterise $\widetilde{\alpha}$ as a Fourier series in three dimensions [16]

$$\widetilde{\alpha}^{(\theta,\tau)}(\mathbf{x}) = \sum_{\mathbf{n}} \theta_{\mathbf{n}} e^{2\pi i \mathbf{k_n} \cdot \mathbf{x}} \,, \tag{10}$$

where π (only in this equation!) is the mathematical constant. The index vector $\mathbf{n}$, wave vector $\mathbf{k_n}$, and location vector $\mathbf{x}$ are

$$\mathbf{x} := (\bar{t}_c, \bar{t}_w, \pi) \qquad n_c \in \{-N_c, \dots, N_c\} \tag{11}$$

$$\mathbf{n} := (n_c, n_w, n_\pi) \qquad n_w \in \{-N_w, \dots, N_w\} \tag{12}$$

$$\mathbf{k_n} := \left(\sigma_c n_c, \sigma_w n_w, \tfrac{n_\pi + \tau}{12}\right) \; n_\pi \in \{-6, \dots, 6\} \,. \tag{13}$$

N_c and N_w allow to independently control the smoothness (or bandwidth) of $\widetilde{\alpha}$ for the center and width dimension, respectively; $\tau \in \mathbb{Z}_{12}$ represents the transposition of the prototype (see below); and $\sigma = 1 - \frac{1}{2N}$ is a scaling factor. Scaling is required because we do not want $\widetilde{\alpha}$ to have periodic boundaries conditions in the time dimensions. The Nyquist frequency of the unscaled function is $2N$, the scaling factor thus stretches the function such that a critical fraction of $\frac{1}{2N}$ is moved out of the interval $[0,1]$. This is not relevant for the pitch dimension because the space of pitch classes is inherently periodic and, moreover, we have a complete discrete Fourier series that allows to represent any function exactly. As $\widetilde{\alpha}$ is real-valued, $\theta_{\mathbf{n}}$ and $\theta_{-\mathbf{n}}$ are complex conjugates and (due to the properties of the discrete Fourier transform) all coefficients with $n_\pi = \pm 6$ are real-valued. We can thus store the parameters θ in a real-valued array of dimensions $(2N_c + 1, 2N_w + 1, 12)$.

As $\widetilde{\alpha}$ (and thus α) are periodic in the pitch dimension, the Fourier representation can be understood as a transposition-invariant formulation of a prototype. When creating a concrete instance of the prototype, τ needs to be specified and defines a specific transposition by inducing a corresponding phase shift through the cyclic pitch class space.

3.2 Mixture Model

In Section 3.1 we defined prototypes that have a point-wise Dirichlet distribution (Definition 4) and adjustable parameters θ. We will now build a transposition-invariant mixture model using these prototypes. The overall structure of the model is shown in Figure 3 as a graphical model [17] and will be explained in detail below.

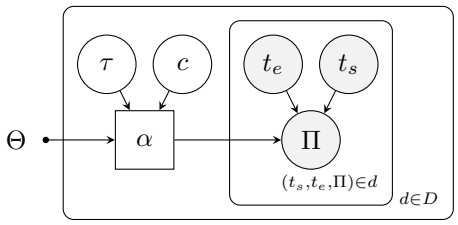

Figure 3. Graphical representation of our mixture model. Θ are the prototype parameters; c and τ the piece-specific cluster index and transposition; α the deterministically generated prototype instance; and $\Pi = \mathfrak{S}(t_s, t_e)$ the pitch scape values at intervals $[t_s, t_e]$ (see text for more details).

Our model is similar to classical topic models for corpora [18–20] with two nested levels. Each piece (or document) d in the data set D is generated independently from a specific prototype with parameters $\theta = \Theta_c$ and transposition τ (outer plate) and for a specific piece, each point Π in its pitch scape is generated independently (inner plate). [1]

3.2.1 Inference

We want to find parameters Θ^* that minimise the cross-entropy (i.e. maximise the likelihood) of our data D

$$\Theta^* = \operatorname{argmin}_\Theta -\tfrac{1}{|D|} \log p(D \,|\, \Theta) \,, \tag{14}$$

where

$$\log p(D \,|\, \Theta) = \sum_{d \in D} \log \sum_{c,\tau} p(d \,|\, \alpha^{(\Theta_c,\tau)})\, p(c)\, p(\tau) \,, \tag{15}$$

is the data log-likelihood with the latent variables c and τ being marginalised out. The prior terms $p(c)$ and $p(\tau)$ are assumed to be constant so that a priori no specific prototype or transposition is preferred. We use

$$\log p(d \,|\, \alpha) = \frac{1}{|d|} \sum_{(t_s,t_e,\Pi)\in d} \log \operatorname{Dir}\big(\Pi; \alpha(t_s, t_e)\big) \tag{16}$$

to approximate the piece likelihood (6) based on a finite number of uniform samples. Marginalising out c and τ also readily yields the normalisation factor for the cluster and transposition probability for a piece

$$p(c, \tau \,|\, d) \propto p(d, c, \tau) = p(d \,|\, \alpha^{(\Theta_c,\tau)})\, p(c)\, p(\tau) \,. \tag{17}$$

The optimal parameters Θ^* can be found by performing gradient descent on the cross-entropy (14).

[1] The assumption of different points in the pitch scape being generated independently is obviously incorrect, which is common to all topic models and the reason why they are not well suited to generate coherent data (e.g. text or music). In fact, in a pitch scape the values at different locations are highly correlated and would ideally be modelled as a single continuous latent function. One approach to achieve this are Gaussian processes (GPs) [21]. However, GPs are computationally expensive and GPs for multi-class classification have complex kernel functions and require approximations of the analytically intractable posterior distribution [21–23]. As we are primarily interested in extracting the mean pitch scape (corresponding to the GP prior), which represents a specific prototype, we therefore chose the simpler approach of defining prototypes as a point-wise Dirichlet distribution.

3.2.2 Hierarchical Clustering

Training the mixture model on a data set allows to perform unsupervised clustering with a fixed number of clusters. However, our motivation is a comparison with the prototypes described in the music theory literature. Instead of choosing a fixed number of clusters, we are rather interested in how clusters split hierarchically from more generic prototypes to more specific ones. We therefore take a hierarchical top-down clustering approach.

We start by training a single prototype on the whole corpus and perform a binary split of this cluster by using it to initialise a mixture of two prototypes, while adding minimal noise (10^{-8}) to the parameters θ to allow the clusters to properly split. This procedure is then recursively and *separately* applied to the resulting prototypes. To this end, the probability $p(d \,|\, c')$ of a piece d to fall into the parent cluster c' is used as a weight in (15) when training the two child clusters. This ensures a clear assignment between parent and child clusters and implies that only pieces that fell into the parent cluster influence the children.

After establishing a hierarchy of prototypes in this way, we perform a joint refinement of the resulting clusters. To this end, *all* final child clusters are combined in a single model while lifting the parent-specific piece weights. This serves a two-fold purpose. First, the prototypes may be sharpened as interactions between the clusters can now be exploited. Second, it acts as a sanity check for the established hierarchy: If the child clusters remain stable in the refinement phase, this indicates consistency of the hierarchical splitting.

4. EVALUATION

4.1 Experimental Setup

The model was implemented in `PyTorch` [24] and the parameters Θ were optimised via gradient descent using the Adam optimiser [25]. The "warm start" with pre-initialised clusters was realised by using a small initial learning rate (10^{-5}) to allow for the mean and variance estimators (internals of the Adam optimiser) to stabilise before reaching the normal learning rate (10^{-3}).

We trained our model on a corpus of 155 Baroque pieces in MIDI format by Johann Sebastian Bach (84%: WTK I/II; Brandenburgisches Konzert No. 5; Inventions and Sinfonias), Georg Friedrich Händel (4%: HWV 264, Movement 2, 4, 9, 10, 11, and 13; HWV 435), and Domenico Scarlatti (12%: Sonatas), see Appendix C for a complete list. As opposed to later periods with an increasing amount of chromaticism, the modulation plans of Baroque pieces are expected to more closely conform to the respective prototypes. Each piece was sampled by choosing interval start and end points on a uniform time grid of $n = 50$ points, resulting in $n(n-1)/2 = 1225$ samples per piece.

Hierarchical clustering was performed up to depth 3 (8 final clusters) with subsequent refinement (see Section 3.2.2). The hierarchy and final prototypes are shown in Figure 4 and discussed below.

4.2 Results and Discussion

The root cluster, which was trained on the entire corpus, represents a generic diatonic prototype. It does not unambiguously belong to a particular mode, being classified as minor based on 'Albrecht' profiles (as in Figure 4) and major based on 'Temperley' profiles. This is also reflected in the separate pitch-scape plots (Table 1 in Appendix B), which have strong weights for the entire pentatonic segment of the line of fifths (C–G–D–A–E). This can be interpreted as confirming the view that Baroque music is fundamentally diatonic.

The initial split results in a clear separation of major and minor keys, with cluster (0) and all its descendants being globally classified as minor pieces while (1) and its descendants are classified as major. From now on we can see a pronounced weight on the tonic pitch class (C for major, A for minor) at the beginning and end of a piece in the separate pitch-scape plots. This split into major and minor prototypes again is an important finding that confirms these two modes to be dominant in Baroque music.

4.2.1 Prototypes in Minor

The next split of the minor cluster separates the two most common prototypes. Prototypical minor-mode pieces are assumed to either modulate to the key of v (the dominant) or to the key of III (relative major) before returning to i (tonic), which corresponds to (0,0) and (0,1) and their descendants, respectively. Note that the cluster (0,1,0) also has a strong tendency to modulate to III, which becomes more apparent when using 'Temperley' profiles.

The i–v–i modulation plan of cluster (0,0,0) and (0,0,1) is one of the two standard prototypes for minor pieces. In (0,0,1), the v is more pronounced and the middle section also has a certain tendency to modulate to III, possibly even including a short VII passage (see 'Temperley' profiles). This corresponds to a i–III–(VII)–v–III–i modulation plan, which is a common subtype of the i–v–i prototype that features two fifth-related, modally distinct key pairs: i–v and III–VII.

Cluster (0,1,0) and (0,1,1) both fall under the general i–III–i prototype. The (0,1,0) cluster has a less pronounced III, which may be partly due to the III being at different locations in the corresponding pieces, thus leading to smoothing/averaging. According to 'Krumhansl' profiles, there is a tendency for modulation to v in the middle section and 'Temperley' profiles classify larger parts of the middle section as III. Cluster (0,1,1) has an additional modulation to the iv after the III, possibly with a short return to the i in between (again this could also be an effect of averaging over multiple pieces), representing the common subtype i–III–(i)–iv–i.

4.2.2 Prototypes in Major

Major pieces are generally assumed to modulate to V before going back to I. However, this general prototype can be elaborated in different ways. For the split of the major cluster (1), we see a very pronounced I–V–I prototype on the left with (1,0) and its child (1,0,0).

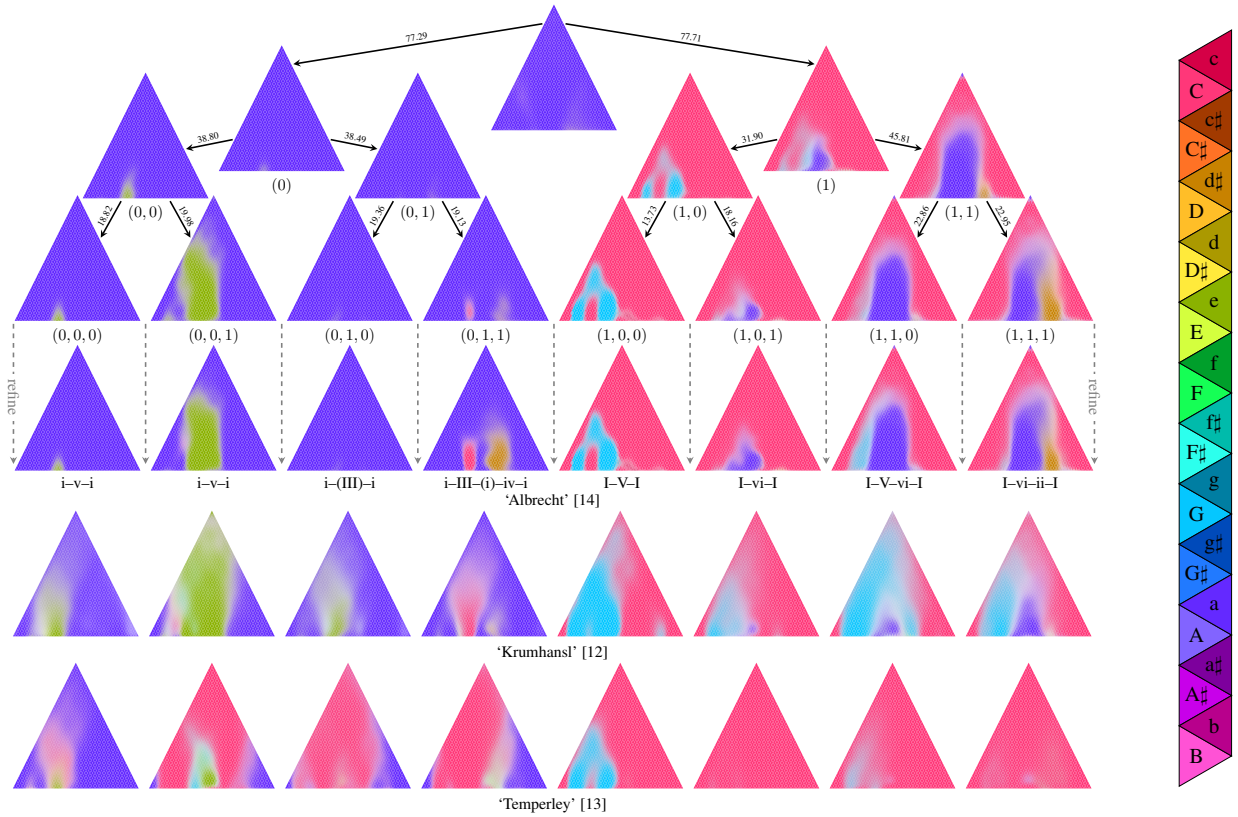

Figure 4. Results of hierarchical clustering with subsequent refinement. To visualise the prototypes, the transposition parameter τ was fixed to minimise the accidentals of the diatonic root cluster. The corresponding absolute keys are shown in the chromatic colour scale (right). However, only relative keys (scale degrees) bear interpretable meaning as the prototypes are inherently transposition-invariant. Prototypes are labelled with a hierarchical index; the final prototypes (after refinement) are labelled with the corresponding modulation plan in Roman numeral notation; numbers on the arrows indicate the number of pieces falling into the respective cluster. Key estimates for colouring are computed using 'Albrecht' [14] templates; the final prototypes are repeated using 'Krumhansl' [12] and 'Temperley' [13] templates to improve interpretability. For better disambiguation, Figure 5 and Figure 6 in Appendix A show chromatic and fifth-based colouring in comparison.

The remaining three clusters all belong to one of the most common elaborations of the I–V–I prototype with an additional vi (relative minor) passage after the V. The V passage is most clearly pronounced in the (1,1,0) cluster, to a lesser extent in the (1,1,1) and even less in the (1,0,1) cluster (see especially the 'Krumhansl' profiles). Notably, (1,1,1) has an additional ii passage after the vi.

The I–vi–I and the I–vi–ii–I cluster taken separately do not contradict expectations from music theory, but due to the missing V they are less typical than the other prototypes so far. However, when being combined with the I–V–vi–I cluster, theses clusters form the very common subtype I–V–vi–ii–I [1]. This is typical in Baroque music but also in modern Pop music, where on the chord-level this sequence is known as "the four chord song" (optionally with a IV as an equivalent pre-dominant replacement of the ii). This combination of multiple prototypes suggests the existence *prototype sub-spaces*.

5. CONCLUSION

To address the problem of modelling and automatic retrieval of prototypical modulation plans from a corpus of musical pieces, a probabilistic Bayesian model of transposition-invariant prototypes was introduced. This model was based on a novel hierarchical *pitch scape* representation of the musical content. We learned prototypical modulation plans from a corpus of Baroque pieces, empirically confirming common prototypes postulated in music theory. Extending the conventional music theoretical concepts, we found that continuous *prototype sub-spaces* can be generated as the superposition of multiple prototypes.

Our approach relies on minimal prior assumptions, works on simple pitch data and delivers robust results while being scalable to large data sets. It can therefore be applied to model, analyse and discover hierarchical key structures and prototypes in a wide range of musical styles and genres, including diachronic studies of musical form and syntax in Western classical music, the influence of style- and composer-specific elements, and the investigation of modulation plans in other genres such as Jazz, Pop and Rock music. Therefore, our approach is suited for numerous applications and contributes a valuable method for music information retrieval.

6. ACKNOWLEDGMENTS

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